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Fixed Point Theorem for the Poincaré Group

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Abstract

The fixed points for an arbitrary proper Poincaré transformation are found by exploiting the homomorphism between the Lorentz group and $SL(2, C)$.

Recently there has been considerable interest in determining fixed points for various physical situations (Atkinson, 1968a, 1968b, 1969, 1970; Lovelace, 1967; Kupsch, 1969; Warnock, 1969). In line with this work, we have determined the fixed points for an arbitrary proper finite Poincaré transformation. Our results represent an extension of the work by Synge (1971) in which he gives an existence proof for fixed points for an arbitrary infinitesimal Poincaré transformation.

Rather than work with the 4×4 matrices of the Lorentz group, we exploit the two to one homomorphism between $SL(2, C)$, the group of 2×2 complex matrices with determinant unity, and the Lorentz group. Thus we benefit from the considerable simplification that accrues from using only 2×2 matrices. The connection between these two groups is as follows (Joos, 1962; Waerden, 1932): If x is any four-vector, there corresponds to it a 2×2 hermitian matrix

$$X = \begin{bmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{bmatrix} \tag{1}$$

such that

$$\det X = x^2 \tag{2}$$

and that each proper Lorentz transformation

$$x_\mu \to x_\mu{}' = \Lambda_{\mu\nu} x_\nu \tag{3}$$

corresponds to a transformation

$$A^\dagger X A = X', \qquad \det A = 1 \tag{4}$$

where A is a 2×2 unimodular matrix which is unique up to its sign. A simple derivation of the explicit formulae expressing Λ in terms of A and vice versa is given in the Appendix.

In the 2×2 notation a Poincaré transformation

$$x_\mu \to x_\mu{}' = \Lambda_{\mu\nu} x_\nu + t_\mu \tag{5}$$

corresponds to the transformation

$$A^\dagger XA + T = X', \qquad T^\dagger = T \tag{6}$$

The fixed point condition is clearly obtained by setting $X' = X$ in (6). Let us exclude the trivial case $A = \pm 1$ for which every point is fixed if $T = 0$ and there are no fixed points if $T \neq 0$. We then have the following theorem:

Theorem: Let

$$A^\dagger XA + T = X \tag{7}$$

be the fixed point equation for the Poincaré group in $SL(2, C)$ notation.

Let $a = tr A$. Then the solutions of equation (7) are as follows:

$a = a^*$: There is a unique solution

$$X = \frac{Q - Q^\dagger}{a^* - a}, \qquad Q = T(A - a) \tag{8}$$

$a = a^*$, $Q \neq Q^\dagger$: There is no solution.
$a = a^*$, $Q = Q^\dagger$: Solutions exist, but are not unique. They are of the form

$$X = \frac{1}{4k^2}(BQ + QB^\dagger) + i(B^\dagger W - WB) \tag{9}$$

where W is any hermitian matrix and

$$B = A - \frac{a}{2}, \qquad 2k^2 = \operatorname{tr} B^\dagger B \tag{10}$$

Here $(BQ + QB^\dagger)/4k^2$ is a particular solution of (7), and $i(B^\dagger W - WB)$ is a solution of the homogeneous equation ($T = 0$).

The space spanned by the homogeneous solution $i(B^\dagger W - WB)$ is two-dimensional and has signature (+,–), (0,–), and (–,–) respectively for $b^2 > 0$, $b^2 = 0$ and $b^2 < 0$ where

$$B^2 = \left(\frac{a^2}{4} - 1\right)\mathbb{1} \equiv b^2\,\mathbb{1} \tag{11}$$

and the Lorentz transformation A is therefore a pure rotation, a pure 'null' transformation, and a pure boost in the three respective cases.

The given translation T is orthogonal to the homogeneous space $i(B^\dagger W - WB)$ for all values of b^2. The particular solution $(BQ + QB^\dagger)/4k^2$ is linearly independent of the homogeneous space $i(B^\dagger W - WB)$ but in general it is not orthogonal to it. However, for $b^2 \neq 0$ the orthogonal part can be projected out and turns out to be

$$X_\pi = \frac{1}{4b^2}(B^\dagger Q + QB) \tag{12}$$

We turn now to the proof of these statements.

Proof: The proof hinges on the fact that the characteristic equation for any 2×2 matrix M is quadratic and of the form

$$M^2 = M \operatorname{tr} M - \det M \tag{13}$$

Applying this result to the matrix A, we obtain

$$A^2 = aA - \mathbb{1}, \qquad a = \operatorname{tr} A \tag{14}$$

and hence

$$A^{-1} = a - A \tag{15}$$

Then multiplying the fixed point equation (7) by A^{-1} on the right, we reduce it to

$$A^\dagger X + XA - aX = Q, \qquad Q = T(A - a) \tag{16}$$

which is linear in A. To solve this equation, the trick is to note that $A^\dagger X + XA$ is self-adjoint whereas the other terms are not. Hence we can eliminate $A^\dagger X + XA$ by subtracting the adjoint equation from (16) with the result that

$$(a^* - a)\, X = Q - Q^\dagger \tag{17}$$

This equation has the unique solution (8) for X in the case $a \neq a^*$. Inserting this solution of (17) into the original fixed point equation (7) or (16) we see that it is identically satisfied. Thus we have the following result: For $a \neq a^*$ there is always a solution to the fixed point equation. It is unique and is given by (8). From (17) we also obtain the result that for $a = a^*$ there is no solution unless $Q = Q^\dagger$.

Let us now assume that $a = a^*$ and $Q = Q^\dagger$. For real a, we see that (16) can be reduced to

$$B^\dagger X + XB = Q, \qquad B = A - \frac{a}{2} \tag{18}$$

Note that B is essentially the matrix A with its trace removed. We then note that all possible solutions of the fixed point equation can be expressed as the sum of one particular solution and all possible solutions of the *homogeneous* fixed point equation (i.e. the fixed point equation with $T = 0$).

We now obtain a particular solution for $a = a^*$ and $Q^\dagger = Q$. For this purpose we first return to the characteristic equation (13) and by setting $M = B$ and $M = B + B^\dagger$ obtain the relations

$$B^2 = -\det B . \mathbb{1} = \left(\frac{a^2}{4} - 1\right)\mathbb{1} \equiv b^2 \mathbb{1} \tag{19}$$

and

$$BB^\dagger + B^\dagger B = [\det B + \det B^\dagger - \det(B + B^\dagger)]\,\mathbb{1} \equiv 2k^2 \mathbb{1} \tag{20}$$

where we have used the fact that B and $B + B^\dagger$ are traceless. Note that b^2 is not definite but that $k^2 > 0$. Second, we note that since

$$T(A - a) = T(B - (a/2))$$

we have

$$Q^\dagger = Q \Leftrightarrow B^\dagger T = TB \Leftrightarrow B^\dagger Q = QB \tag{21}$$

Then using (20) (21) and (19) (21) respectively it is easy to verify that

$$X_P = \frac{1}{4k^2}(BQ + QB^\dagger) \tag{22}$$

is a solution of the fixed point equation for all b^2, and that

$$X_\pi = \frac{1}{4b^2}(B^\dagger Q + QB) \tag{23}$$

is a particular solution for $b^2 \neq 0$. Since the particular solutions (22) and (23) were not obtained deductively from the fixed point equation, the question arises in this case as to whether these solutions are unique. To answer this question we look for solutions X_H to the *homogeneous* fixed point equation which from (16) may be written in the form

$$B^\dagger X_H + X_H B = 0 \tag{24}$$

From this equation and (19) we see that

$$X_H = i(B^\dagger W - WB) \tag{25}$$

where W is any hermitian 2×2 matrix, is a solution. We now wish to show that (25) is the most general solution of (24). In other words, we wish to show that (25) always has a solution for W when X_H satisfies (24). To show this we let $C = -iB$ and rewrite (24) and (25) in the form

$$C^\dagger W + WC = X_H \tag{26}$$

where

$$C^\dagger X_H = X_H C \tag{27}$$

But these two equations are just the original equations we had for the non-homogeneous equation if we make the substitutions ($B \to C$, $W \to X$,

$X_H \to Q$). Since we have displayed in (22) a particular solution for $X(B, Q)$, a solution to (24) and (25) for $W(C, X_H)$ is clearly

$$W = \frac{1}{4k^2}(CX_H + X_H C^\dagger) = \frac{-i}{4k^2}(BX_H - X_H B^\dagger) \tag{28}$$

One can, of course, also verify directly that W in (28) satisfies (25) when X_H satisfies (24). Thus (25) is in fact the most general solution to the homogeneous fixed point equation and $X = X_P + X_H$ is the most general solution to the original inhomogeneous fixed point equation.

We conclude by giving a geometrical interpretation of the space $i(B^\dagger W - WB)$ of solutions to the homogeneous fixed point equation. For this purpose we consider the cases $b^2 \neq 0$ and $b^2 = 0$ separately. When $b^2 \neq 0$, it is convenient to renormalise the traceless matrix B so that its determinant becomes unity. Thus we define

$$R = \frac{i}{b}B = \frac{i}{\sqrt{\left(\frac{a^2}{4} - 1\right)}}\left(A - \frac{a}{2}\right), \qquad \det R = 1 \tag{29}$$

R is then a Lorentz transformation with respect to which the Minkowski space of all hermitian matrices W has the decomposition

$$W = W_+ + W_- \tag{30}$$

where

$$W_\pm = \tfrac{1}{2}(W \pm R^\dagger WR) \tag{31}$$

since

$$R^\dagger W_\pm R = \pm W_\pm \tag{32}$$

R has the property that it splits Minkowski space into an invariant and a non-invariant part. It is easy to see that the invariant part W_+ contains the timelike vector $1 + R^\dagger R$. This shows that R is a rotation, and hence that W_+ is a two-dimensional space (spanned by $1 + R^\dagger R$ and the three-dimensional axis of rotation). On the other hand, the non-invariant part of Minkowski space simply changes sign under the action of R. This shows that R is actually a rotation through an angle π, and that W_- is a spacelike two-dimensional space which is orthogonal to W_+. From the property (32) which characterises the spaces $W_\pm$ and from the definitions of X_π and X_H and the condition (21) for T, we can easily see that

$$\begin{array}{lll} X_\pi, T \in W_+, & X_H \in W_-, & \text{for } b^2 > 0 \\ X_\pi, T \in W_-, & X_H \in W_+, & \text{for } b^2 < 0 \end{array} \tag{33}$$

The particular solution X_P in (22) is not an eigenstate of R and for this reason we prefer the particular solution X_π for $b^2 \neq 0$.

For $b^2 = 0$, the geometrical interpretation is most easily obtained by noting that in this case the four hermitian matrices

$$BB^\dagger, \qquad B^\dagger B, \qquad B^\dagger + B, \qquad i(B^\dagger - B) \tag{34}$$

are non-zero and linearly independent. Since $\det B = 0$ and $\det(B^\dagger \pm B) = -(B^\dagger \pm B)^2 = \mp 4k^2$, the first two of these vectors are null and the last two are spacelike.

It is then easy to see that the homogeneous solution X_H, the given vector T and the special solution X_P lie in the following two-spaces

$$\begin{aligned} X_H &\in \{i(B^\dagger - B), B^\dagger B\} \\ T &\in \{B^\dagger + B, B^\dagger B\} \\ X_P &\in \{B^\dagger + B, BB^\dagger\} \end{aligned} \tag{35}$$

where in each case the bracket denotes the two-space spanned by the vectors inside it. In fact, the first two results follow immediately from equations (24) and (21) respectively. The last result follows by noting from (22) that X_P satisfies the relation $BX_P = X_P B^\dagger$ and since this is just the relation satisfied by T, except that $B \rightleftarrows B^\dagger$, the space is the same as that for T except that $B \rightleftarrows B^\dagger$. All of the spaces in (35) have signature (0,–).

Appendix

The explicit formulae connecting Λ and A are

$$\Lambda_{\mu\nu} = \tfrac{1}{2}\,\mathrm{tr}\,(A^\dagger \sigma_\nu A \sigma_\mu) \tag{A1}$$

and

$$A = \pm D/\sqrt{\det D}, \qquad D = \Lambda_{\nu\mu}\sigma_\mu \eta \sigma_\nu \tag{A2}$$

where

$$\sigma_\mu = (\mathbb{1}, \boldsymbol{\sigma}) \tag{A3}$$

$\boldsymbol{\sigma}$ being the Pauli-matrices, and η is any 2×2 matrix such that $D \neq 0$. If possible $\eta = 1$. Note that since $\det A = 1$, $D \neq 0$ implies $\det D \neq 0$.

Proof: From equation (1) we have

$$X = x_\mu \sigma_\mu \tag{A4}$$

and hence from equations (2) and (3) we have

$$A^\dagger \sigma_\mu A = \Lambda_{\nu\mu}\sigma_\nu \tag{A5}$$

To obtain (A1) from this equation, we multiply to the right by σ_τ and take the trace. Since the σ_μ are trace-orthogonal we obtain

$$\mathrm{tr}\, A^\dagger \sigma_\mu A \sigma_\tau = 2\Lambda_{\tau\mu} \tag{A6}$$

which is clearly equivalent to (A1).

To obtain the inverse formula (A2) the trick is to note that if for any matrix N we define

$$N' = \sum_{\mu=1}^{4} \sigma_\mu . N\sigma_\mu = \sum_{\mu=1}^{4} \sigma_\mu N \sigma_\mu^{-1}, \tag{A7}$$

we have

$$
\begin{aligned}
\sigma_\lambda N' &= \sum_{\mu=1}^{4} (\sigma_\lambda \sigma_\mu) N \sigma_\mu^{-1} \\
&= \sum_{\mu=1}^{4} (\sigma_\lambda \sigma_\mu) N (\sigma_\lambda \sigma_\mu)^{-1} \sigma_\lambda \\
&= \sum_{\tau=1}^{4} (\varepsilon \sigma_\tau) N (\varepsilon \sigma_\tau)^{-1} \sigma_\lambda, \qquad \varepsilon = \pm 1, \pm i \\
&= \sum_{\tau=1}^{4} \sigma_\tau N \sigma_\tau^{-1} \sigma_\lambda \\
&= N' \sigma_\lambda
\end{aligned} \tag{A8}
$$

It follows that N' is a multiple of the unit matrix, and hence that

$$N' = (2 \operatorname{tr} N) \mathbb{1} \tag{A9}$$

We now multiply equation (A5) to the left by $\sigma_\mu \eta$ and sum over μ. From (A9) we obtain

$$2 \operatorname{tr} (\eta A^\dagger) A = \Lambda_{\nu\mu} \sigma_\mu \eta \sigma_\nu = D \tag{A10}$$

and hence using $\det A = 1$ we have (A2), as required.

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